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Advanced Linear Algebra (MA 409)
Problem Sheet - 8

Composition of Linear Transformations and Matrix Multiplication

1. Label the following statements as true or false. In each part, V,W, and Z denote vector spaces with ordered (finite) bases α, β, and γ, respectively; $T : V \to W$ and $U : W \to Z$ denote linear transformations; and A and B denote matrices.

   (a) $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$.

   (b) $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ for all $v \in V$.

   (c) $[U(w)]_\beta = [U]_\alpha^\beta [w]_\beta$ for all $w \in W$.

   (d) $[I_V]_\alpha = I$.

   (e) $[T^2]_\alpha^\beta = ([T]_\alpha^\beta)^2$.

   (f) $A^2 = I$ implies that $A = I$ or $A = -I$.

   (g) $T = L_A$ for some matrix A.

   (h) $A^2 = O$ implies that $A = O$, where O denotes the zero matrix.

   (i) $L_{A+B} = L_A + L_B$.

   (j) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$.

2. Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \quad \text{and} \quad U(a + bx + cx^2) = (a + b, c, a - b).$$

   Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

   (a) Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$ directly. Verify that $[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta$.

   (b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

3. Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$\beta = \{1, x, x^2\},$$

   and

$$\gamma = \{1\}.$$

   Compute the following vectors:

(a) $[T(A)]_\alpha$, where $A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$ and $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ defined by $T(A) = A^t$.

(b) $[T(f(x))]_\alpha$, where $f(x) = 4 - 6x + 3x^2$ and $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$.

(c) $[T(A)]_\gamma$, where $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $T : M_{2\times 2}(F) \to F$ defined by $T(A) = tr(A)$.

(d) $[T(f(x))]_\gamma$, where $f(x) = 6 - x + 2x^2$ and $T : P_2(\mathbb{R}) \to \mathbb{R}$ defined by $T(f(x)) = f(2)$.

4. Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

5. Let A be an $n \times n$ matrix. Prove that A is a diagonal matrix if and only if $A_{ij} = \delta_{ij} A_{ij}$ for all i and j.

6. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ (the zero operator) if and only if $R(T) \subseteq N(T)$.

7. Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

   (a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

   (b) Prove that if UT is onto, then U is onto. Must T also be onto?

   (c) Prove that if U and T are one-to-one and onto, then UT is also.

8. Let A and B be $n \times n$ matrices. Prove that $tr(AB) = tr(BA)$ and $tr(A) = tr(A^t)$.

9. (a) Suppose that z is a (column) vector in F^p. Prove that Bz is a linear combination of the columns of B. In particular, if $z = (a_1, a_2, \ldots, a_p)^t$, then show that

$$Bz = \sum_{j=1}^{p} a_j v_j.$$

   (b) Extend (a) to prove that column j of AB is a linear combination of the columns of A with the coefficients in the linear combination being the entries of column j of B.

   (c) For any row vector $w \in F^m$, prove that wA is a linear combination of the rows of A with the coefficients in the linear combination being the coordinates of w.
   Hint: Use properties of the transpose operation applied to (a).

   (d) Prove the analogous result to (b) about rows: Row i of AB is a linear combination of the rows of B with the coefficients in the linear combination being the entries of row i of A.

10. Let M and A be matrices for which the product matrix MA is defined. If the jth column of A is a linear combination of a set of columns of A, prove that the jth column of MA is a linear combination of the corresponding columns of MA with the same corresponding coefficients.

11. Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear.

   (a) If $rank(T) = rank(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$.

   (b) Prove that $V = R(T^k) \oplus N(T^k)$ for some positive integer k.

12. Let V be a vector space. Determine all linear transformations $T : V \to V$ such that $T = T^2$.

   Hint: Note that $x = T(x) + (x - T(x))$ for every x in V, and show that $V = \{y : T(y) = y\} \oplus N(T)$.

13. Using only the definition of matrix multiplication, prove that multiplication of matrices is associative.

14. For an incidence matrix A with related matrix B defined by $B_{ij} = 1$ if i is related to j and j is related to i, and $B_{ij} = 0$ otherwise, prove that i belongs to a clique if and only if $(B^3)_{ii} > 0$.

15. Use the above exercise to determine the cliques in the relations corresponding to the following incidence matrices.

$$(a) \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad (b) \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

16. Let A be an incidence matrix that is associated with a dominance relation. Prove that the matrix $A + A^2$ has a row [column] in which each entry is positive except for the diagonal entry.

17. Prove that the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   corresponds to a dominance relation. Use the above exercise to determine which persons dominate [are dominated by] each of the others within two stages.

18. Let A be an $n \times n$ incidence matrix that corresponds to a dominance relation. Determine the number of nonzero entries of A.

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